PRESSURE PROPAGATION IN VISCOELASTIC MEDIA DURING MOTION IN TUBES OF ELASTOVISCOUS MATERIAL

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The process of pressure propagation is studied in viscoelastic media during their flow in thin-walled tubes of viscoelastic Maxwell and Foigt materials.

A number of studies [1-3] have been dedicated to questions of pressure propagation in Newtonian and non-Newtonian media with equilibrium rheological parameters, moving in elastic and viscoelastic tubes. However, it is known that many real media (melted and dissolved polymers, conglomerate materials with a polymer binder [4], etc.) manifest viscoelastic properties under dynamic conditions. Moreover, recent studies [5, 6] have shown that many highly viscous petroleum mixtures with a high content of resin, asphalt, and paraffin components, as well as clay and cement solutions containing polymer additives will manifest relaxation properties under certain conditions. It should be noted that nonrigid tubes (well columns, tubes with a layer of resin, asphaltenes, or paraffin deposited on the wall, tubes of polymer material, etc.) also have viscoelastic properties which must be considered in describing pressure propagation.

In the general case, viscoelastic media undergo noninfinitesimal deformations. However, as was shown in [7-9], the theory of linear viscoelasticity can be employed with the following assumptions:

1) in the case of small perturbations which lead to small deformations of the medium;

2) in cases where the relaxation functions and medium parameters are not dependent on the value of the deformation;

3) in simple shear, where noninfinitesimal and infinitesimal deformations coincide as to the character of the change produced in the deformation tensor.

Using these assumptions, [10] derived differential equations for the non-steady-state motion of a generalized viscoelastoplastic medium in tubes of viscoelastic material. In the present study we will examine the process of pressure propagation in flow of such a medium within tubes of viscoelastic material, since consideration of the relaxation properties of both the moving medium and the tube material upon transient regimes is of both theoretical and practical interest in many engineering situations.

1. We will consider non-steady-state motion of a relaxation viscoelastic medium, described by the rheological equation $\theta\dot{\sigma} + \sigma = \mu(\dot{\gamma} + \lambda\ddot{\gamma})$ in thin-walled tubes of a viscoelastic Maxwell ($\dot{\varepsilon} = \dot{\tau}/G + \tau/\eta$) and Foigt ($\tau = G\varepsilon + \eta\dot{\varepsilon}$) material.

The problem reduces to solution of a system of differential equations

$$\theta\frac{\partial^2 P}{\partial t\partial x} + \frac{\partial P}{\partial x} = -\rho_0\left(\theta\frac{\partial^2 W}{\partial t^2} + (1+2a\lambda)\frac{\partial W}{\partial t} + 2aW\right),$$
$$\frac{\partial P}{\partial t} + \frac{1}{\tau_1}\frac{c^2}{c_\tau^2}P = -c^2\rho_0\frac{\partial W}{\partial x} \tag{1}$$

for the Maxwell tube and

$$\theta\frac{\partial^2 P}{\partial t\partial x} + \frac{\partial P}{\partial x} = -\rho_0\left(\theta\frac{\partial^2 W}{\partial t^2} + (1+2a\lambda)\frac{\partial W}{\partial t} + 2aW\right),$$

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$$\frac{c^2}{c_l^2}\tau_2\frac{\partial^2 P}{\partial t^2}+\frac{\partial P}{\partial t}=-c^2\rho_0\left(\frac{\partial W}{\partial x}+\tau_2\frac{\partial^2 W}{\partial t\partial x}\right)\tag{2}$$

for the Foigt tube, where

$$2a=\frac{8\mu}{\rho_0 R^2};\ c^2=\frac{K_l}{\rho_0(1+2RK_l/G\delta_0)};$$

$$c_\tau^2=\frac{G}{\rho_0};\ c_l^2=\frac{K_l}{\rho_0}.$$

Elimination of the velocity from Eqs. (1), (2) produces the following differential equations in pressure:

$$c^2\left(\theta\frac{\partial}{\partial t}+1\right)\frac{\partial^2 P}{\partial t^2}=\frac{\partial U}{\partial t}+\beta_1 U,\tag{3}$$

$$c^2\left(\theta\frac{\partial}{\partial t}+1\right)\left(\tau^2\frac{\partial}{\partial t}+1\right)\frac{\partial^2 P}{\partial x^2}=\beta_2\frac{\partial^2 U}{\partial t^2}+\frac{\partial U}{\partial t}.\tag{4}$$

Here

$$U=\theta\frac{\partial^2 P}{\partial t^2}+(1+2a\lambda)\frac{\partial P}{\partial t}+2aP;$$

$$\beta_1=\frac{1}{\tau_1}\frac{c^2}{c_\tau^2},\quad \beta_2=\tau_2\frac{c^2}{c_l^2}.$$

2. We will consider non-steady-state motion of the relaxation medium in a viscoelastic medium, with the pressure being a harmonic function of time with specified frequency $\omega$ at the beginning of the tube. At times sufficiently later than the initial time, the effect of initial conditions on the pressure distribution will be negligible.

We must then find the solution of Eqs. (3), (4) which satisfies the boundary condition

$$P(0,\ t)=P_0 e^{i\omega t}.\tag{5}$$

The solution of Eqs. (3), (4) satisfying boundary condition (5) has the form

$$P(x,\ t)=P_0 e^{i\omega t+\alpha x},\tag{6}$$

where

$$\alpha=\sqrt{k_1+i\omega k_2}/c;\ k_1=(R_1+\theta\omega^2 S_1)/(1+\theta^2\omega^2);\ k_2=(S_1-\theta R_1)/(1+\theta^2\omega^2);$$
$$R_1=(2a\beta_1-(\beta_1\theta+1+2a\lambda)\omega^2);\ S_1=2a+(1+2a\lambda)\beta_1-\theta\omega^2$$

in the case of Maxwell tube material and

$$k_1=((1-\theta\tau_2\omega^2)R_2+(\theta+\tau_2)\omega^2 S_2)/Q;\ k_2=((1-\theta\tau_2\omega^2)S_2-(\theta+\tau_2)R_2)/Q;\ R_2=\omega^2(\theta\beta_2\omega^2-1-2a\lambda-2a\beta_2);$$
$$S_2=2a-\omega^2(\theta+(1+2a\lambda)\beta_2);\ Q=(1+\theta^2\omega^2)(1+\tau_2^2\omega^2)$$

in the case of the Foigt model.

Separating the real and imaginary components of $\alpha$ and choosing that root which has a negative real component, we write solution (6) in the form

$$P=P_0 e^{-\xi\frac{x}{c}}e^{i\omega\left(t-\nu\frac{x}{c}\right)},\tag{7}$$

where $\xi=\sqrt{(k_3+k_1)/2}$, $\nu=k_2/2\xi$; $k_3=\sqrt{k_1^2+\omega^2 k_2^2}$. The parameter $\xi$ characterizes the damping, and the parameter $\nu$ is the delay or phase shift of the boundary value of the pressure harmonic.

First we will study the effect of the medium relaxation parameters $\theta$ and $\lambda$ upon the distribution of the parameters $\xi$ and $\nu$ over frequency $\omega$.

The dependence of $\xi$ and $\nu$ on frequency $\omega$ is shown in Fig. 1. The curves 2 of Fig. 1a correspond to the larger $\theta$ value. The dashed curves indicate the frequency distributions of the corresponding parameters for a viscous liquid [1]. It is evident from Fig. 1a that over some low frequency interval the values of $\xi$ for a relaxation liquid are lower than for a viscous one, while with increase in $\theta$ in this interval the $\xi$ values increase although the in-

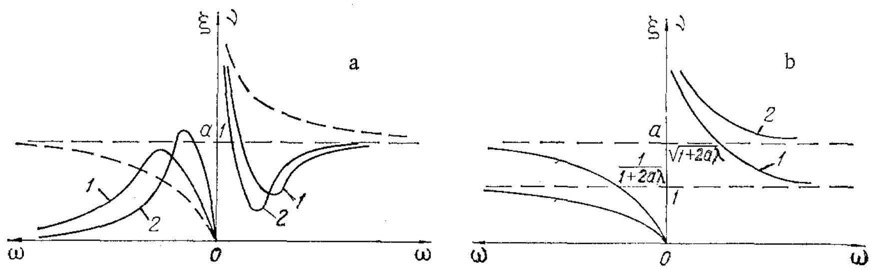


Fig. 1. Quantities $\xi$ and $\nu$ vs frequency for motion of relaxation medium in elastic tube: a) $\theta \neq 0$, $\lambda = 0$; b) $\theta = 0$, $\lambda \neq 0$ .

terval itself narrows. With increase in frequency, $\xi$ asymptotically vanishes. This, for example, for $a = 1\ \sec^{-1}$, $\theta = 5$ sec at $\omega = 2\ \sec^{-1}$ the parameter $\xi = 0.02\ \sec^{-1}$.

Analysis of Fig. 1b reveals that the presence of the relaxation parameter $\lambda$ does not change the qualitative pattern of $\xi$ and $\nu$ distribution over frequency, as compared to a viscous liquid (curves 1 and 2, respectively), although the actual numerical values can differ significantly.

In the case where both relaxation parameters $\theta$ and $\lambda$ are nonzero, for a given frquency the values of the parameters $\xi$ and $\nu$ are larger than those of a viscous liquid at $\theta < \lambda$ and smaller for $\theta > \lambda$. At $\theta = \lambda$ the damping and delay parameter values coincide with the corresponding ones of a viscous liquid.

We will now consider the effect of viscoelastic tube wall properties on the parameters $\xi$ and $\nu$. In the case of the Maxwell model $\xi$ increases monotonically with increase in frequency from $\gamma_1 = \sqrt{2a\beta_1}$ to $\gamma_2 = a + \beta_1/2$, these being the geometric and arithmetic means of the coefficients $2a$ and $\beta_1$, while the value of $\gamma_2/\gamma_1$ decreases from 1 to unity. Also, at $\beta_1 = 2a$ the parameters $\xi$ and $\nu$ are independent of frequency. The character of the $\xi$ and $\nu$ distributions over frequency as a function of the coefficients $2a$ and $\beta_1$ is shown in Fig. 2.

In the case of a Foigt tube the viscoelastic properties of the walls decrease the delay parameter $\nu$ and at high frequencies $\nu$ asymptotically approaches the value $c/c_l$. Figure 3 shows the frequency distribution of $\nu$ . Curve 1 corresponds to an elastic tube, and curves 2 and 3 to presence of a parameter $\tau_2$, with curve 3 being for the larger $\tau_2$. At high frequencies the damping parameter asymptotically approaches the value $ac/c_l + (1 - c^2/c_l^2)/2\beta_2$. Small $\tau_2$ produce a higher damping of the pressure harmonic, while at large $\tau_2$ the damping parameter decreases.

The case in which a relaxation medium described by a rheological equation with $\lambda \neq 0$ and $\theta = 0$ moves in a viscoelastic tube is qualitatively no different from the cases studied above in which pressure boundary harmonics propagate in a viscous liquid. One need only change all expressions in which the coefficients $a$ and c appear, using the substitutions $a_* = a/(1 + 2a\lambda)$ and $c_* = c/\sqrt{1 + 2a\lambda}$.

If $\theta \neq 0$ in the rheological model, the value of the damping parameter at higher frequencies asymptotically approaches $\beta_1/2$ for a Maxwell tube, i.e., an adaptive superposition of the two effects occurs. Liquid relaxation tends to compensate viscous resistance corresponding to the coefficient $2a$, while tube relaxation, as in the motion of a viscous liquid, increases the damping parameter $\xi$ at high frequencies by an amount $\beta_1/2$. At low frequencies, it is essentially the viscoelastic properties of the tube walls which manifest themselves.

3. We assume that initially the flow and pressure in the entire tube occupying the semispace $x \geqslant 0$ is constant and equal to zero, and at the moment $t = 0$ at the boundary $x = 0$ a pressure $\varphi(t)$ is applied.

The solution of the problem reduces to solution of differential equation (3) for the case of a Maxwell tube, and Eq. (4) for a Foigt tube. The initial and boundary conditions for the problem specified have the form

$$\frac{\partial^i P}{\partial t^i}(x,\ 0) = 0,\ i = 0,\ 1,\ 2,\ 3, \tag{8}$$

$$P(0,\ t) = \varphi(t),\ P(\infty,\ t) = 0,\ t > 0.$$

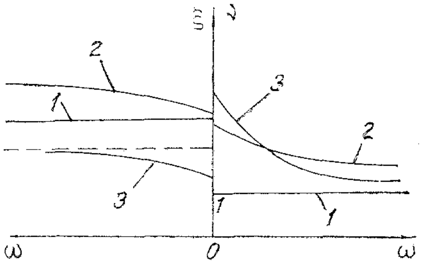

Fig. 2

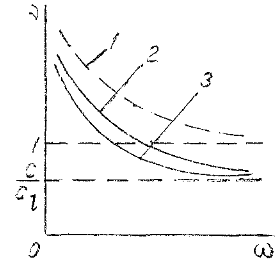

Fig. 3

Fig. 2. Quantities $\xi$ and $\nu$ vs frequency for viscous liquid motion in a tube of Maxwell material: 1) $\beta_1 = 2a$; 2) $\beta_1 > 2a$; 3) $\beta_1 < 2a$.

Fig. 3. Quantity $\nu$ vs frequency for viscous liquid motion in a tube of Foigt material.

Taking the Laplace transform of Eqs. (3), (4), (8), we obtain

$$c^2(\theta s+1)\frac{d^2P^*}{dx^2}=(s+\beta_1)(\theta s^2+(1+2a\lambda)s+2a)P^*, \tag{9}$$

$$c^2(\theta s+1)(\tau_2 s+1)\frac{d^2P^*}{dx^2}=s(\beta_2 s+1)(\theta s^2+(1+2a\lambda)s+2a)P^*, \tag{10}$$

$$P^*(0,\ s)=\varphi^*(s),\ P^*(\infty,\ s)=0, \tag{11}$$

where

$$P^*(x,\ s)=\int_0^\infty e^{-st}P(x,\ t)\,dt,\quad \varphi^*(s)=\int_0^\infty e^{-st}\varphi(t)\,dt.$$

The solutions of Eqs. (9) and (10) with boundary condition (11) have the form:

$$P^*(x,\ s)=\varphi^*(s)\exp\left(-\frac{x}{c}\sqrt{\frac{(\theta s^2+(1+2a\lambda)s+2a)(s+\beta_1)}{\theta s+1}}\right), \tag{12}$$

$$P^*(x,\ s)=\varphi^*(s)\exp\left(-\frac{x}{c_l}\sqrt{\frac{(\theta s^2+(1+2a\lambda)s+2a)(s+1/\beta_2)s}{(\theta s+1)(s+1/\tau_2)}}\right). \tag{13}$$

Equations in the original variables may be obtained from Eqs. (12), (13) just as in [2]. The perturbation propagation velocity is independent of the relaxation parameters of the moving medium. In the case of a Maxwell tube material this velocity is the same as in an elastic tube with viscous liquid flow and is equal to c. For flow of a relaxation medium in a tube of Foigt material, the propagation velocity is higher than for flow in an elastic tube, and equal to $c_l$.

The equations in the original variables stemming from Eqs. (12), (13) are quite cumbersome in the general case. Below we will only analyze special cases and the asymptotic behavior of these solutions.

In a number of cases to be considered below, we can write the original equations in the form

$$P(x,\ t)=\begin{cases}0 \quad \text{at}:\ 0\leqslant t\leqslant x/c_*,\\ e^{-\frac{B}{2c_*}x}\varphi(t-x/c_*)-\dfrac{x\sqrt{D}}{c_*}\displaystyle\int_{x/c_*}^{t}\varphi(t-m)\times\\ \times e^{-\frac{bm}{2}}\dfrac{J_1\left(\dfrac{\sqrt{D}}{c_*}\sqrt{m^2-x^2/c_*^2}\right)}{\sqrt{m^2-x^2/c_*^2}}dm \quad \text{at}\ t\geqslant x/c_*,\end{cases} \tag{14}$$

where $D = d - b^2/4$. For example, for a Maxwell model, the original stemming from Eq. (12) has the form of Eq. (14) with $\beta_1 = 1/\theta(c_* = c,\ b = (1 + 2a\lambda)/\theta,\ d = 2a/\theta)$, and $\theta = \lambda(c_* = c,\ b = 2a + \beta_1,\ d = 2a\beta_1)$. For the Foigt model the original from Eq. (13) has the form of Eq. (14) at $\theta = \lambda$ and $2a = 1/\tau_2(c_* = c_l,\ b = 1/\beta_2,\ d = 0)$.

When a step in pressure is specified at the beginning of the tube, the value of the perturbation propagation front depends significantly on the system parameters. Analysis of the special cases indicated above show that Maxwell tube properties encourage more rapid perturbation damping, while Foigt properties produce more rapid propagation and damping of the pressure step as compared to an elastic tube.

Below we will consider the asymptotic behavior of the solutions of Eqs. (3), (4) for propagation of a pressure step.

In the case of Maxwell tube material, if we assume that $\theta s \ll 1$ only the relaxation properties of the tube have an effect on front propagation. The pressure at the front will then be:

$$P = P_0 \exp(-(a + \beta_1/2)\, t), \tag{15}$$

where $P_0$ is the initial pressure step.

It is evident from Eq. (15) that Maxwell tube properties encourage greater damping of perturbations in comparison to an elastic tube, for which the pressure is described by the expression $P_0 \exp(-at)$.

Equation (15) may be treated in two ways. On the one hand, it describes the front pressure at large times. On the other, it defines the front pressure at arbitrary time for low values of the parameter $\theta$.

In the case of Foigt tube material at large times ($s \ll \min\{1/\theta,\ 1/\tau_2\}$) the relaxation properties of the tube material and the moving medium have practically no effect on perturbation propagation. At short time intervals, both in a Maxwell tube ($s \gg \max\{1/\theta,\ \beta_1\}$) and in a Foigt tube ($s \gg \max\{1/\theta,\ 1/\beta_2\}$) the perturbation magnitude depends only on the relaxation properties of the moving medium. In these cases the pressure has the form

$$P = P_0 \exp(-((1 + 2a\lambda)/2\theta)\, t).$$

4. For a finite tube the boundary conditions for the case of a pressure step at the beginning of the tube may be written

$$\begin{cases} \dfrac{\partial^i P}{\partial t^i}(x,\ 0) = 0,\ i = 0,\ 1,\ 2,\ 3,\ t \leqslant 0, \\ P(0,\ t) = P_0,\ P(l,\ t) = 0,\ t > 0. \end{cases} \tag{16}$$

Solutions of Eqs. (3) and (4) with boundary condition (16) can be obtained by using Laplace transforms and expanding the latter in eigenfunctions of the corresponding homogeneous boundary problem:

$$P = P_0 (l - x)/l - (2P_0/l) \sum_{n=1}^{\infty} \frac{\sin \mu_n x}{\mu_n} f_n(t), \tag{17}$$

$$P = P_0 (l - x)/l - (2P_0/l) \sum_{n=1}^{\infty} \frac{\sin \mu_n x}{\mu_n} g_n(t), \tag{18}$$

where $\mu_n = \pi n/l$, and $f_n(t)$ and $g_n(t)$ are the originals corresponding to

$$f_n^*(s) = \frac{(s + \beta_1)(\theta s^2 + (1 + 2a\lambda)s + 2a)}{s((\theta s^2 + (1 + 2a\lambda)s + 2a)(s + \beta_1) + c^2\mu_n^2(\theta s + 1)}, \tag{19}$$

$$g_n^*(s) = \frac{(\beta_2 s + 1)(\theta s^2 + (1 + 2a\lambda)s + 2a)}{s(\theta s^2 + (1 + 2a\lambda)s + 2a)(\beta_2 s + 1) + c_m^2\, \mu_n^2(\theta s + 1)(\tau_2 s + 1)}. \tag{20}$$

The originals of Eqs. (19), (20) can be calculated for specified parameter values by expansion in simple fractions:

$$f_n(t) = A_{no} + \sum_{i=1}^{3} A_{ni} \exp(-s_{ni} t), \tag{21}$$

$$g_n(t) = \sum_{i=0}^{3} B_{ni} \exp(-s_{ni} t), \tag{22}$$

where $s_{ni}$ are the roots of the denominators of Eqs. (19), (20); $A_{ni}$ and $B_{ni}$ are constants dependent on the eigennumbers $\mu_n$ and parameters of Eqs. (3), (4). Depending on the form of the roots $s_{ni}$ the function $f_n(t)$ consists of the sum of constants and three damping exponentials or exponentially damping harmonic components, while $g_n(t)$ is the sum of four exponential or exponentially damping components. It can be shown that $f_n(t)$ and $g_n(t)$ contain no purely harmonic components for any n.

The stationary pressure distribution for the Foigt model of the tube material is linear, as in the case of an elastic tube. For the Maxwell model, the stationary pressure distribution has the form

$$P_{\infty} = P_0 \operatorname{sh} \sqrt{\frac{2a}{\tau_1}} \frac{l-x}{c_{\text{T}}} \Big/ \operatorname{sh} \sqrt{\frac{2a}{\tau_1}} \frac{l}{c_{\text{T}}}. \tag{23}$$

Thus, the relaxation properties of the medium moving in an elastoviscous tube do not affect the stationary pressure distribution, just as in tubes with Foigt properties. However tube walls with Maxwell properties do exert an effect in the stationary flow regime.

## NOTATION

$\sigma$ and $\gamma$, stress and deformation in medium rheological model; $\tau$ and $\varepsilon$, stress and deformation in tube material rheological model; $\mu$, viscosity of medium; $\eta$, viscosity of tube material; G, modulus of elasticity of tube material; $K_l$, modulus of volume compression of liquid; $\theta$, $\lambda$, $\tau_1$ and $\tau_2$, relaxation times; $\rho_0$, density of medium, R, tube radius; $\delta_0$, tube wall thickness; P, pressure; W, average flow velocity over tube section.

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